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ENTROPY CONJECTURE FOR CONTINUOUS MAPS OF NILMANIFOLDS

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ABSTRACT

In 1974 Michael Shub asked the following question [29] : When is the topological entropy of a continuous mapping of a compact manifold into itself is estimated from below by the logarithm of the spectral radius of the linear mapping induced in the cohomologies with real coefficients? This estimate has been called the Entropy Conjecture (EC). In 1977 the second author and Michał Misiurewicz proved [23] that EC holds for all continuous mappings of tori. Here we prove EC for all continuous mappings of compact nilmanifolds. Also generalizations for maps of some solvmanifolds and another proof via Lefschetz and Nielsen numbers, under the assumption the map is not homotopic to a fixed points free map, are provided.

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1. Introduction.

Definition 1.1: For a compact metric space $(X\,,d)$ a continuous mapping $f: X \to X$, $\epsilon > 0$, a positive integer n and a set $Q \subset X$

a) Q is called (n,ϵ)-separated if for any two distinct points $x, y \in Q$

$$\max_{0 \le j \le n} d(f^j(x), f^j(y)) \ge \epsilon.$$

b) Put $r(f,\epsilon) := \limsup_{n\to\infty} \frac{1}{n} \log \max\{\mathrm{Card} Q : Q \text{ is an } (n,\epsilon)\text{-separated subset of } X\}$,

c) The **topological entropy** $\mathbf{h}(f)$ is a nonnegative real number or ∞ defined

$$\mathbf{h}(f) = \lim_{\epsilon\to 0} r(f,\epsilon) = \sup_{\epsilon\to 0} r(f,\epsilon).$$

Now assume that X is a compact smooth manifold M of dimension m.

Definition 1.2: Let $H^*(f) : H^*(M;\mathbb{R}) \to H^*(M;\mathbb{R})$ be the linear map induced by f on the cohomology space $H^*(M;\mathbb{R}) := \bigoplus_0^m H^i(M;\mathbb{R})$ of M with real coefficients.

By $\mathrm{sp}\,(f)$ we denote the spectral radius of $H^*(f)$, which is a homotopy invariant by definition.

In 1974 Michael Shub asked, [29, p. 36], the extent to which the following inequality holds.

(EC) $$\log \mathrm{sp}\,(f) \le \mathbf{h}(f).$$

Shub noted that it holds for a set of $C^r, r \ge 1$, diffeomorphisms dense in the C^0-topology. From this time the (EC) estimate has been usually called the **Entropy Conjecture**, abbr. **EC**. Roughly speaking, the entropy conjecture is an estimate from below of a "global" (algebraic) ingredient of the topological entropy of f, not removable under perturbations of f within its, say, homotopy class.

Remark that without any additional assumption on M the conjecture is false. Consider for example the mapping of the sphere S^2

$$f(x,y) = \big(2x(\mathrm{mod} 2\pi), \pi \sin(1/2(y+\pi/2)) - \pi/2\big),$$

where $0 \le x \le 2\pi$ is the longitude and $-\pi/2 \le y \le \pi/2$ is the altitude on the 2-sphere. Then $\log \mathrm{sp}\,(f) = \log \deg(f) = \log 2$ and the entropy, "concentrated" at the set of non-wandering points, the south and north poles, thus is equal to 0. For more details see [29], pp. 37–38.

Next, Anthony Manning proved that $\mathbf{h}(f)$ is bounded from below by $\log \operatorname{sp} H^1(f)$, see [20]. It was noticed in [3], [5] and [15] that this lower bound could be improved to the estimation by the growth of the endomorphism of the fundamental group of M, induced by f.

In 1977 the second author and Misiurewicz proved [23] that EC holds for all continuous mappings f on M being tori. Anatole Katok conjectured [15] that EC holds for all continuous mappings for M being a manifold with the universal cover homeomorphic to $\mathbb{R}^m$.

To complete this history recall that M. Shub conjectured [29], Conjecture 3a, that EC holds for all C^1-mappings on all compact manifolds. The answer occurred to be positive for all C^∞ maps, see [33], and also for all C^1 maps if we consider only the m-dimensional cohomologies, see [22]. In the latter case EC takes the form $\log|\deg(f)| \leq \mathbf{h}(f)$.

In this paper we make a first, to the best of our knowledge, progress after 27 years, for continuous maps, going beyond tori, to all nilmanifolds. The methods in our main proof go back to 1970-ties, [21, 6] and an unpublished version of [23]. However we need to overcome troubles coused by noncommutativity of the groups G, Γ, see below, and non-hyperbolicity of $H^*(f)$. For comments on difficulties in a general situation see Remarks 2.13 and 4.8.

In Section 4, we provide a different proof, via asymptotic Lefschetz and Nielsen numbers, provided f is not homotopic to a fixed point free map.

We call a discrete subgroup Γ of a Lie group G such that the homogeneous space Γ is compact a **uniform lattice**. We shall prove the following.

Theorem 1.3: *The estimate (EC) holds for all continuous maps $f : M \to M$ of a compact manifold M which is a quotient of a connected, simply-connected nilpotent Lie group by a uniform lattice.*

Let us discuss the assumptions in this theorem and some related notions. We include solvmanifolds in the discussion since at some points of the paper the nilpotency is not needed.

a) A homogeneous space $M = G/H$ of a connected solvable or nilpotent Lie group G is called a **solvmanifold** or **nilmanifold** respectively. If G is simply-connected and the subgroup $H = \Gamma$ is discrete M is called a **special solvmanifold** (cf., [17, p. 5]).

For a connected nilpotent Lie group G every compact nilmanifold $M = G/H$ is special (cf., [8, II Chapter 4, Theorem 1.2]). Therefore Theorem 1.3 concerns all compact nilmanifolds.

b) A Lie group G is said to be **exponential** if for any X in its Lie algebra $\mathcal{G}$ the nonzero eigenvalues of the operator ad_X are not purely imaginary. Any exponential Lie group is solvable. Connected, simply-connected exponential Lie groups are characterized by the fact that the exponential map $\exp : \mathcal{G} \to G$ from the Lie algebra $\mathcal{G}$ to G, is a diffeomorphism (cf., [28], [32, Chapter 2, §5.3]).

c) By **endomorphism** we mean in this paper a group endomorphism, continuous (hence smooth) in the case of a Lie group, or a Lie algebra endomorphism. A Lie group G is said to be **of type (R)**, from **real**, if for any X in its Lie algebra $\mathcal{G}$ all the eigenvalues of the operator ad_X are real. For any Lie group G of type (R) and a uniform lattice $\Gamma \subset G$ any endomorphism of Γ extends to an endomorphism of G (cf. [28], [32, Chapter 2, §5.3]). Let us call the latter property **rigidity**. Since in a nilpotent Lie algebra $\mathcal{G}$ for every X the operator ad_X is nilpotent, a nilpotent Lie group is of type (R).

d) Every solvmanifold $M = G/H$, such that G is exponential or of type (R) is called an **exponential solvmanifold** or **solvmanifold of type (R)** respectively.

e) The nilradical $\mathcal{N}$ in a Lie algebra $\mathcal{G}$ is its maximal nilpotent ideal. The nilradical N in a Lie group G is its maximal connected nilpotent normal subgroup. If $\mathcal{G}$ is the Lie algebra of G, then $N = \exp \mathcal{N}$.

Note finally that, even without assuming that G is exponential, every connected simply-connected solvable Lie group is homeomorphic to $\mathbb{R}^m$, thus contractible, (cf., [26] Introduction 1.9). So a special solvmanifold $M = G/\Gamma$ is a $K(\pi, 1)$ space (cf., [31]) with the fundamental group Γ. Consequently the set of free homotopy classes of self maps is in one to one correspondence with the set of conjugacy classes of endomorphisms of Γ, [31, Chapter 8.1, Theorem 11].

The assumption that G is of type (R), hence the rigidity, allows to find in the free homotopy class $[f]$ of a continuous mapping $f : M \to M$, a mapping ϕ_f whose lift to G is an endomorphism Φ_f, extending an endomorphism of Γ in the conjugacy class corresponding to $[f]$.

We call Φ_f an endomorphism associated to f. (Note that Φ_f and ϕ_f can exist for particular f even in absence of the rigidity property.)

The proof of Theorem 1.3 extends easily (see Remark 2.12) to the proof of the following

THEOREM 1.4: *The estimate (EC) holds for all continuous maps $f : M \to M$ of a compact solvmanifold of type (R) $M = G/\Gamma$ with G connected, simply-connected and Γ a uniform lattice, provided the images of sufficiently high iterates of the endomorphism $\Phi_f : G \to G$ are contained in the nilradical of G.*

The proof of Theorem 1.3 in Section 2 has two steps. We denote by

$$\bigwedge D\Phi_f(e) := \bigoplus_{l=0}^{m} \bigwedge^l D\Phi_f(e)$$

a linear operator induced by $D\Phi_f(e)$ in the exterior algebra $\bigwedge^*(\mathbb{R}^m) = \bigoplus_{l=0}^m \bigwedge^l \mathbb{R}^m$ of $\mathcal{G}$ considered as the linear space $\mathbb{R}^m$.

It is useful to note a standard fact that

$$\log \operatorname{sp}\big(\bigwedge D\Phi_f(e)\big) = \log \prod_j |\lambda_j| \tag{1}$$

the product over all eigenvalues of $D\Phi_f(e)$ counted with multiplicities, of absolute value greater than 1 provided that $\operatorname{sp}(D\Phi_f(e)) > 1$. The case $\operatorname{sp}(D\Phi_f(e)) \leq 1$ leads to the obvious inequality $\mathbf{h}(f) \geq 0$ and we will not discuss it.

In the first step we use a direct argument (Theorem 2.1) to show that

$$\log \operatorname{sp}(f) \leq \log \operatorname{sp}\big(\bigwedge D\Phi_f(e)\big). \tag{2}$$

In the second step, the main one, we prove that for such a map we have the estimate

$$\log \operatorname{sp}\big(\bigwedge D\Phi_f(e)\big) \leq \mathbf{h}(f). \tag{3}$$

A hardest technical point, Lemma 2.7, is moved to Section 3. We substantially use the nilpotency there, exploring tha assumption that G is "almost commutative".

In Section 4, we point out that (2) can be obtained in different ways. For a continuous self-map of a compact special solvmanifold the inequality $\operatorname{sp}(f) \leq \operatorname{sp}\big(\bigwedge D\Phi_f(e)\big)$ is a direct consequence of the Nomizu and Hattori theorem [24] identifying $H^*(M;\mathbb{R})$ with the cohomology of the Lie algebra $\mathcal{G}$. Another way is to identify $\operatorname{sp}(D\Phi_f(e))$ with the spectrum of an integer matrix A_f called linearization of f, assigned to f via the descending central tower of ideals of the nilradical of $\mathcal{G}$, [17].

Recall that the descending central tower of ideals in a nilpotent Lie algebra $\mathcal{N}$ is the sequence

$$\{0\} = \mathcal{G}_k \lhd \mathcal{G}_{k-1} \lhd \mathcal{G}_{k-2} \lhd \cdots \lhd \mathcal{G}_1 \lhd \mathcal{G}_0 = \mathcal{N}\,, \tag{4}$$

where $\mathcal{G}_j = [\mathcal{N}, \mathcal{G}_{j-1}]$. Analogously, the descending central tower of normal subgroups in a nilpotent Lie group N whose Lie algebra is $\mathcal{N}$ is the sequence $\{G_j = \exp(\mathcal{G}_j),\ j = k, \dots, 0\}$.

Next, in Section 4, we show how (3) Theorem 1.3, can be obtained via Lefschetz and Nielsen asymptotic numbers, provided that $f : M \to M$ is not homotopic to a fixed point free map, referring to [4], [1], [17] and [12]. In conclusion we obtain

THEOREM 1.5: *The estimate (EC) holds for any continuous map $f : M \to M$ of a special solvmanifolds of type (R) which is not homotopic to a fixed point free map.*

At the end of Section 4 we give other comments and final remarks.

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2. Proof of Theorem 1.3.

To prove the estimate (2) we use the following theorem.

THEOREM 2.1: *Let $M = G/\Gamma$ be a compact homogeneous space of a connected Lie group G by a uniform lattice Γ, and $\phi : M \to M$ the factor map induced by an endomorphism $\Phi : G \to G$ preserving Γ. Then*

$$\log \operatorname{sp}(\phi) \le \log \operatorname{sp}\big(\textstyle\bigwedge D\Phi(e)\big).$$

Proof. We shall use de Rham cohomologies. Fix a right invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on G, i.e., a metric such that every differential Dg of the right multiplication $h \mapsto hg$ is an isometry. Such a metric can defined starting from an arbitrary scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{G} = T_eG$ by $\langle w_1, w_2\rangle_{T_g} := \langle v_1, v_2\rangle_{T_e}$ for $w_1 = Dg(v_1), w_2 = Dg(v_2) \in T_gG$. By the right invariance this metric induces a Riemannian metric on $M = G/\Gamma$. This metric induces a metric and thus a norm on each fiber of the exterior power $\bigwedge^j T_gG$ and consequently on $\bigwedge^j T_xM$, $x \in M$, (just the volume of the parallelopiped $v_1 \wedge \cdots \wedge v_j$) and on the fibers of dual bundles $\bigwedge^j T_g^*G$, and $\bigwedge^j T_x^*M$ respectively. A norm $\| \ \|_{T_x}$ in $\bigwedge^j T_x^*M$ gives a norm of any differential j-form ν defined as $\|\nu\| := \sup_{x\in M} \|\nu(x)\|$.

Every differentiable map $\psi : M \to M$ induces a linear map of the spaces of differentiable forms, denoted by $\bigwedge(D\psi)^*$. Note that for any closed continuous differential form j-form η on M and every $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\| \bigwedge(D\psi^n)^*(\eta)\| \\
&\quad = \sup\left\{ (|\eta(D\psi^n(v_1) \wedge \cdots \wedge D\psi^n(v_j))| : \begin{array}{l} x \in M, v_1, \cdots, v_j \in T_xM, \\ \|v_1 \wedge \cdots \wedge v_j\| = 1 \end{array} \right\} \\
&\quad \le \|\eta\| \cdot \sup\left\{ \|D\psi^n(v_1) \wedge \cdots \wedge D\psi^n(v_j)\| : \begin{array}{l} x \in M, v_1, \cdots, v_j \in T_xM, \\ \|v_1 \wedge \cdots \wedge v_j\| = 1 \end{array} \right\}.
\end{aligned}$$

For any closed j-form η and its cohomology class $[\eta]$ one defines the norm $\|[\eta]\| := \inf\{\|\nu\| : \nu \in [\eta]\}$ (cf., [10]).

For $\phi : M \to M$ being the factor to M of an endomorphism Φ of G we have

$$\log \mathrm{sp}\,(\phi) \le \sup_j \lim_{n\to\infty} \sup_{x\in M} \frac{1}{n} \log \left\| \bigwedge^j D(\phi^n)(x) \right\| = \log \mathrm{sp}\,\big(\bigwedge D\Phi(e)\big). \tag{5}$$

The first inequality of (5) is a consequence of the above considerations and definitions, and the definition of spectral radius of a map (Def. 1.2). The latter equality of (5) follows from the fact that $\| \bigwedge^j D(\phi^n)(x)\|$ does not depend on x. Indeed, for any $h \in G$ denote by H_1, and H_2 the right multiplications by h, and $\Phi(h)$ respectively. From the equality $\Phi(x) = \Phi(xh^{-1})\Phi(h)$ it follows that

$$\|D\Phi(h)\| = \|DH_2 \circ D\Phi(e) \circ DH_1^{-1}(h)\| = \|D\Phi(e)\|,$$

since H_1 and H_2 are isometries by our definition of the Riemannian metric. The same holds for exterior powers and iterates of ϕ. ∎

Proof of the estimate (2). Since the spectral radius is a homotopy invariant, the supposition follows from Theorem 2.1. ∎

The rest of Section 2 (and Section 3) will be devoted to the estimate 3. Namely we shall prove the following

THEOREM 2.2: *For a continuous map $f : M \to M$ for a compact nilmanifold $M = G/\Gamma$ and an endomorphism $\Phi_f : G \to G$ associated to f*

$$\log \mathrm{sp}\left(\bigwedge D\Phi_f(e)\right) \leq \mathbf{h}(f)\,.$$

For a given endomorphism $\Phi : G \to G$ of a Lie group G denote by E^s, E^c, E^{cs}, E^u, E^{cu} the linear subspaces of its Lie algebra $T_eG = \mathcal{G}$ being the direct sums of generalized eigenspaces (related to Jordan cells) corresponding to eigenvalues of $D\Phi(e)$ with absolute values smaller than 1, equal to 1, not exceeding 1, larger than 1, larger and equal to 1. The superscripts s, c, cs, u, and cu abbreviate: **stable**, **central central-stable**, **unstable** and **central-unstable** respectively. Let E^0 denote the generalized eigenspace corresponding to 0. By definition $E^0 \subset E^s$. Let also E^+ be a linear subspace corresponding to all nonzero eigenvalues i.e. $E^0 \oplus E^+ = \mathcal{G}$. Sometimes if we have in mind any of $s, c, cs, u, cu, 0, +$ we write $\#$. Finally it is substantial to note that $D\Phi(e)$ is a Lie algebra endomorphism. As a direct consequence of these definitions we get the following.

PROPOSITION 2.3: *For an endomorphism Φ of a Lie group G, the subspaces E^s, E^c, E^{cs}, E^u, E^{cu} of its Lie algebra $\mathcal{G}$, are Lie subalgebras of $\mathcal{G}$, and E^0 is an ideal. All of them are preserved by the endomorphisms $D\Phi(e)$. Furthermore, E^s is an ideal in E^{cs} and E^u in E^{cu} respectively.*

Proof. To shorten notation we put $D := D\Phi(e)$. Note that

$$E^0 = \{X \in \mathcal{G} : \exists n \in \mathbb{N}, \quad D^n(X) = 0\}.$$

So, for $X \in E^0$, $Y \in \mathcal{G}$, we have $D^n([X,Y]) = [D^nX, D^nY] = [0, D^nY] = 0$. So E^0 is an ideal. More generally, writing E_λ for the direct sum of the kernels of large powers of $D - \lambda I$ and $D - \bar{\lambda} I$ (the subspace spanned by the subspaces corresponding to all the Jordan cells of $\lambda, \bar{\lambda}$) we can easily prove that $[E_\lambda, E_\mu] \subset E_{\lambda\mu}$, see [2, Exercise §4 21a]. This implies by the definition that all $E^\#$ are subalgebras, as direct sums of the appropriate E_λs, since if $E_\lambda, E_\mu \subset E^\#$, then $E_{\lambda\mu} \subset E^\#$. Finally E^s is an ideal in E^{cs} since $|\lambda| < 1$, $|\mu| \leq 1$ implies $|\lambda\mu| < 1$. Analogously we prove that E^u is an ideal in E^{cu}. ∎

Note that we can write a dynamical characterization of the subspaces $E^{\#}$. For example denoting the isomorphism $D|_{E^+}$ by $\tilde{D}$ we can write

$$E^s = \{X \in \mathcal{G} : \limsup_{n\to\infty} \sqrt[n]{\|D^n(X)\|} < 1\},$$

$$E^c = \{X \in E^+ : \lim_{n\to\pm\infty} \sqrt[|n|]{\|\tilde{D}^n(X)\|} = 1\},$$

$$E^u = \{X \in E^+ : \limsup_{n\to\infty} \sqrt[n]{\|\tilde{D}^{-n}(X)\|} < 1\}.$$

So, for example,

$$\|D^n[X,Y]\| = \|[D^nX, D^nY]\| \leq C\lambda^n \|X\| \cdot \|Y\|$$

for an arbitrary λ between $\max\{|\nu| : |\nu| < 1, \nu \in \sigma(D)\}$ and 1, a constant $C > 0$ and all $n \geq 0$, implies that if $X \in E^s, Y \in E^{cs}$, then $[X,Y] \in E^s$.

Consider $G^{\#} = \exp E^{\#}$ for $\# = 0, s, cs, s, u, cu, +$, Lie subgroups of G. By E^0 being an ideal, G^0 is normal (but the other are usually not normal in G). Also, G^u is normal in G^{cu}. We will be interested in the foliations W^s, W^c and W^{cs} being the decompositions of G into the quotiens G^sg, G^cg and $G^{cs}g$. $E^sg, E^cg, E^{cs}g$ are subbundles tangent to these foliations respectively. (Formally we act on $E^{\#}$ with a differential of the right multiplication by g, we shall use however a simplified notation $E^{\#}g$.)

Proposition 2.4: *For an endomorphism Φ of a Lie group G which is connected, simply-connected and exponential,*

$$G^sg = \{h \in G : \limsup_{n\to\infty} \sqrt[n]{\rho(\Phi^n(g), \Phi^n(h))} \leq \lambda^s\}, \tag{6}$$

$$G^ug = \{h \in G^+ : \limsup_{n\to\infty} \sqrt[n]{\rho(\tilde{\Phi}^{-n}(g), \tilde{\Phi}^{-n}(h))} \leq 1/\lambda^u\}, \tag{7}$$

where λ^s is the spectral radius of $D|_{E^s}$, $1/\lambda^u$ is the spectral radius of $(\tilde{D}|_{E^u})^{-1}$, $\tilde{\Phi} = \Phi|_{G^+}$, and

$$G^{cs}g = \{h \in G : \limsup_{n\to\infty} \sqrt[n]{\rho(\Phi^n(g), \Phi^n(h))} \leq 1\}. \tag{8}$$

Proof. The inclusion $\subset$ in (6) follows from

$$\rho(\Phi^n(g), \Phi^n(h)) = \int_0^1 \|D\Phi^n(X)(\exp tX)\|\, dt,$$

where $hg^{-1} = \exp X$, i.e. $(\exp tX)g, 0 \leq t \leq 1$ is a geodesic γ, joining g to h, and from $E^s = \{\limsup_{n\to\infty} \sqrt[n]{\|D^n(X)\|} \leq \lambda^s\}$.

To prove the opposite inclusion (which we do not need in this paper, so we only sketch the proof) write any $x \in G$ as $x = g_1 g_2 g$ for $g_1 \in G^{cu}$, $g_2 \in G^s$, see the notation and comments preceding Lemma 2.6. Then $\rho(\Phi^n(x), \Phi^n(g)) \geq \rho(\Phi^n(x), \Phi^n(g_2 g)) - \rho(\Phi^n(g_2 g), \Phi^n(g))$. The latter term decays exponentially to 0, by (6). Meanwhile, if $g_1 \neq e$, $\rho(\Phi^n(x), \Phi^n(g_2 g)) = \int_0^1 \|D\Phi^n(X)(\exp tX)\| dt \geq \mathrm{Const}\, \xi^n$, where $g_1 = \exp X$ and $X \in E^{cu} \setminus \{0\}$, and ξ is arbitrarily close to 1. The path $\gamma = \exp tX$, $0 \leq t \leq 1$ is the shortest one joining $g_2 g$ to x, since there is only one geodesic (one-parameter group) joining e to g_1 since the group is exponential. The proofs of (7) and (8) are similar. ∎

The plan of the proof of Theorem 2.2. For a given continuous map $f : M \to M$ of a compact nilmanifold $M = G/\Gamma$ let $\tilde{f} : G \to G$ be a lift of f such that $\Phi = \Phi_f$, an endomorphism associated to f, and $\tilde{f}$ are joined by a lift to G of a homotopy between ϕ_f and f. We say that the lift $\tilde{f}$ **corresponds** to Φ_f. To each $\tilde{f}$-trajectory in G we assign a Φ-trajectory in G^u. Next we verify that (n, ϵ)-separated points (trajectories) for Φ in G^u will be assigned to (n, ϵ')-separated points for $\tilde{f}$ and after projection to $M = G/\Gamma$, for f.

This assignments will be done in 3 steps: first mapping by $\tau^{cu} : G \to G^{cu}$ next by $\tau^u : G^{cu} \to G^u$ along quotients (leaves) of the foliations W^s and W^c, respectively, and finally by shadowing of ϵ_n-Φ-trajectories in G^u by Φ-trajectories. Here ϵ_n will be sub-exponentially increasing and x_n is called ϵ_n-Φ-trajectory if $\rho(\Phi(x_n), x_{n+1}) \leq \epsilon_n$.

We start with some general standard facts concerning nilpotent Lie groups.

Denote a default right invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G introduced above by ω. The metric on G induced by any Riemannian right invariant metric ν will be denoted by ρ_ν, the distance from unity $\rho_\nu(g, e)$ by $r_\nu(g)$ and the norm of $v \in T_g G$ by $\|v\|_\nu$. We have $\|[X, Y]\|_\nu \leq C_\nu \|X\|_\nu \|Y\|_\nu$ for a constant C_ν and all $X, Y \in \mathcal{G}$. For $\nu = \omega$ we omit the index ω.

Note that if $g \in G_1$ is a Lie subgroup of G, then $r_\nu(g)$ in G and in the metric restricted to G_1 coincide, so there is no ambiguity of the notation (see the end of proof of Proposition 2.4).

Lemma 2.5: *Let G be a connected simple connected nilpotent Lie group of descending central tower of ideals in its Lie algebra of length k and let r denote the distance from e, as above. Then, for all $g, h \in G$*

$$r(g^{-1} h g) \leq C r(h)(1 + r(g) + \cdots + \frac{1}{(k-1)!} r(g)^{k-1}).$$

Proof.

$$r(g^{-1}hg) = \int_0^1 \|(e^{\mathrm{ad}\,\hat{g}}\hat{h})(g^{-1}(\exp t\hat{h})g)\|dt$$
$$\leq C\|\hat{h}\|\,(1+\|\hat{g}\|+\cdots+\frac{1}{(k-1)!}\|\hat{g}||^{k-1}), \tag{9}$$

where $h = \exp\hat{h}$, $g = \exp\hat{g}$. In the equality the identity $e^{\mathrm{ad}\,\hat{g}}\hat{h} = \mathrm{Ad}_g\hat{h}$ is used. The last inequality follows from the formula $e^{\mathrm{ad}\,\hat{g}} = \sum_{j=0}^{\infty}\frac{1}{j}(\mathrm{ad}\,\hat{g})^j$ and from the nilpotency. ∎

If G_1, G_2 are Lie subgroups of G such that $\dim G_1 + \dim G_2 = m = \dim G$, $G_1 \cap G_2 = \{e\}$, for $g = g_1g_2, g_1 \in G_1, g_2 \in G_2$ we define "projections" τ_i to $G_i, i = 1, 2$ by $\tau_i(g) = g_i$. Note that $g_2 = \tau_2(g) = g_1^{-1}g$ is the intersection point of the quotient (leaf) G_1g with G_2. If $\mathcal{G}_1$, $\mathcal{G}_2$ are Lie subalgebras of a nilpotent (or exponential) Lie algebra $\mathcal{G}$ and $\mathcal{G}_1 \oplus \mathcal{G}_2 = \mathcal{G}$, $\mathcal{G}_1 \cap \mathcal{G}_2 = \{0\}$ then $G_1 = \exp(\mathcal{G}_1)$, $G_2 = \exp(\mathcal{G}_2)$ satisfy the above conditions, every $g \in G$ is of the form g_1g_2 and τ_i are continuous ([26] Preliminaries 1.9).

Let μ denote a modulus of continuity of these functions, namely consider

$$\mu_i(r) := \sup_{g:r(g)\leq r} r(g_i) \quad \text{and} \quad \mu(r) := \max\{\mu_1(r), \mu_2(r)\}.$$

Lemma 2.6: *For any $g, g' \in G$*

$$\rho(\tau_2(g), \tau_2(g')) \leq \mu\Big(C\mu(\rho(g, g'))(1 + r(\tau_1(g)) + \cdots + \frac{1}{(k-1)!}r(\tau_1(g))^{k-1})\Big). \tag{10}$$

If G_2 is a normal subgroup, then

$$\rho(\tau_1(g), \tau_1(g')) \leq \mu(\rho(g, g')), \tag{11}$$

and

$$\rho(\tau_2(g), \tau_2(g')) \leq C\mu(\rho(g, g'))(1 + r(\tau_1(g)) + \cdots + \frac{1}{(k-1)!}r(\tau_1(g))^{k-1}). \tag{12}$$

Proof. Let $g = g_1g_2$, $g' = h_1h_2g$, where $h_1, g_1 \in G_1, h_2, g_2 \in G_2$. Then $\rho(g, g') = r(h_1h_2)$. We have

$$g' = h_1h_2g_1g_2 = h_1g_1(g_1^{-1}h_2g_1)g_2\,, \tag{13}$$

We write $g_1^{-1}h_2g_1 = k_1k_2$ for $k_i \in G_i$, $i = 1,2$ so we can write $\tau_2(g') = k_2g_2$. Finally, note that $\rho(\tau_2(g), \tau_2(g')) = r(k_2)$ and using $r(k_2) \leq \mu(r(g_1^{-1}h_2g_1))$ and applying (9) in Lemma 2.5, we obtain (10).

To prove (11) note that, by the normality of G_2, $g_1^{-1}h_2g_1 \in G_2$, hence $\tau_1(g') = h_1g_1$ Since $\tau_1(g) = g_1$ we get $\rho(\tau_1(g), \tau_1(g')) = r(h_1) \leq \mu(\rho(g,g'))$.

(12) follows directly from (9) since, due to the normality of G_2, we have $k_2 = g_1^{-1}h_2g_1$. ∎

Now we return to the dynamical subgroups.

Lemma 2.7 (Projection to G^{cu}): *For every $\epsilon, \epsilon_1 > 0$ there exist Δ_1, $\Delta_2 > 0$ such that for every ϵ-Φ-trajectory $x_n \in G$, $n = 0, 1, \ldots$ (that is $\rho(\Phi(x_n), x_{n+1}) \leq \epsilon$), for the pair of groups $G_1 = G^s$, $G_2 = G^{cu}$, and respective projections $\tau_1 = \tau^s$ and $\tau_2 = \tau^{cu}$, if $r(\tau_1(x_0)) \leq \epsilon_1$, then*

$$r(\tau_1(x_n)) \leq \Delta_1\,, \tag{14}$$

and

$$\tau_2(x_n) \text{ is a } \Delta_2 - \Phi - \text{trajectory in } G^{cu}. \tag{15}$$

This is the hardest part of the proof of Theorem, since G^{cu} need not be normal in G and we have a trouble with (11). We postpone the proof of it. Note that if G^{cu} were normal, then for $x_{n+1} = h\Phi(x_n)$ the condition $r(h) \leq \epsilon$ would imply by (11) that $\rho(\tau_1(x_{n+1}),\, \tau_1(\Phi(x_n))) \leq \mu(\epsilon)$ hence, roughly, $r(\tau_1(x_{n+1}) \leq \mu(\epsilon) + \lambda^s r(\tau_1(x_n))$, where $\lambda^s < 1$ is the weakest contraction rate in G^s. The latter is less than $r(\tau_1(x_n))$ provided $\mu(\epsilon) < (1-\lambda^s)r(\tau_1(x_n))$. So $r(\tau_1(x_n)) < \Delta_1$ for all n by induction, provided $\mu(\epsilon) < (1-\lambda^s)\Delta_1$ and $r(\tau_1(x_0)) \leq \Delta_1$. The final step, that (14) yields (15), is immediate from (10); the normality is not used.

Lemma 2.8 (Projection to G^u): *Consider $\tau_1 = \tau^c$ and $\tau_2 = \tau^u$ for the pair of groups $G_1 = G^c$ and $G_2 = G^u$ in G^{cu}. Then for every $\epsilon', \epsilon_1' > 0$ and $\xi > 1$, there exists $C \geq 1$ such that for every ϵ'-Φ-trajectory $x_n \in G^{cu}$, $n = 0, 1, \ldots$ such that $r(\tau_1(x_0)) \leq \epsilon_1'$, we have*

$$r(\tau_1(x_n)) \leq C\xi^n \tag{16}$$

and

$$\rho(\tau_2(x_{n+1}), \Phi(\tau_2(x_n))) \leq C\xi^n. \tag{17}$$

Proof. By the definition of E^c there exists $C(\xi) > 0$ such that for every $n = 0, 1, \ldots$, $\|D\Phi^n|_{E^c}(e)\| \leq C(\xi)\xi^n$. By (11) in Lemma 2.6, for $g = \Phi(x_n)$, $g' = x_{n+1}$ we obtain $\rho(\tau_1(\Phi(x_n)), \tau_1(x_{n+1})) \leq \mu(\epsilon')$, hence $r(\tau_1(x_{n+1})) \leq r(\Phi(\tau_1(x_n)) + \mu(\epsilon')$. Composing this for $n = 1, 2, \ldots$ we get, with the use of (8) in Proposition 2.4,

$$r(\tau_1(x_n)) \leq C(\xi)\xi^n r(\tau_1(x_0)) + \sum_{j=0}^{n-1} C(\xi)\xi^j \mu(\epsilon') \leq C(\xi)\epsilon_1'\xi^n + C(\xi)\mu(\epsilon')\frac{\xi^n - 1}{\xi - 1}$$
$$\leq \Big(C(\xi)\epsilon_1' + C(\xi)\mu(\epsilon')\frac{1}{\xi - 1}\Big)\xi^n,$$

which proves (16) with C defined by the expression in the parentheses. (17) follows from (12). First we get the estimate by $C\xi^{kn}$ with a new C, but k does not appear at the end if we start with $\xi^{1/k}$. ∎

Remark 2.9: With a slightly more effort we could prove that the growths in (16) and (17) are polynomial. However we shall not use it in the sequel.

Proposition 2.10: *Let $f : M \to M$ be a map of a compact nilmanifold $M = G/\Gamma$, $\tilde{f} : G \to G$ its lift corresponding to $\Phi = \Phi_f : G \to G$ an associated endomorphism. Then there exists a continuous map $\theta : G \to G^u$ which is "onto", moreover $\theta|_{G^u}$ is onto G^u, and such that*

$$\theta \circ \tilde{f} = \Phi \circ \theta. \tag{18}$$

Moreover, for every $\xi > 1$ there exists $C \geq 1$ such that for all $x \in G^u$, $n \in \mathbb{N}$, $x_n = \tilde{f}^n(x)$

$$\rho(\tau^u \tau^{cu}(x_n), \theta(x_n)) \leq C\xi^n. \tag{19}$$

Proof. Since $\tilde{f}$ corresponds to Φ, namely, $\tilde{f}$ and Φ_f are joined by a lift of a homotopy between f and ϕ_f on M, their distance is bounded by a constant ϵ. We construct $\theta : G \to G^u$ as follows. Let $x \in G$. Then the $\tilde{f}$-trajectory x_n is an ϵ-Φ-trajectory. Hence by Lemmas 2.7 and 2.8 $y_n = \tau^u\tau^{cu}(x_n)$ is an ϵ_n-Φ-trajectory in G^u with $\epsilon_n = C\xi^n$, with $C = C(x)$ depending on $r(\tau_1(x))$ in the decomposition $G = G^s G^{cu}$ and $r(\tau_1(\tau^{cu}(x)))$ in the decomposition $G^{cu} = G^c G^u$, and on ϵ. Note that by Lemmas 2.7 and 2.8 $C(x)$ can be chosen to depend on each of these 3 variables in a monotone way. In consequence $C(x)$ is locally bounded.

We "shadow" this ϵ_n-Φ-trajectory by a Φ-trajectory z_n in G^u given by the formula

$$z_n = \lim_{j\to\infty} (\Phi|_{G^u})^{-j}(y_{n+j})\,. \tag{20}$$

Note that, for λ an arbitrary constant such that $1 < \lambda < \lambda_u$ infimum of the absolute values of the eigenvalues of $D\Phi(e)$ larger than 1, (see (7)) for a constant C_λ, writing $\tilde{\Phi} := \Phi|_{G^u}$, we get, provided $\xi < \lambda$,

$$\begin{aligned} \rho(y_n, z_n) &\le \sum_{j=0}^{\infty} \rho(\tilde{\Phi}^{-j}(y_{n+j}), \tilde{\Phi}^{-(j+1)}(y_{n+j+1})) \\ &\le \sum_{j=0}^{\infty} C_\lambda \lambda^{-(j+1)} \rho(\Phi(y_{n+j}), y_{n+j+1}) \\ &\le \sum_{j=0}^{\infty} C_\lambda \lambda^{-(j+1)} \epsilon_{n+j} \\ &\le C(x) C_\lambda \xi^n \frac{1}{1-\lambda^{-1}\xi}\,. \end{aligned} \tag{21}$$

Observe that $\Phi(z_n) = z_{n+1}$ follows immediately from the definition (20).

Define $\theta(x) = z$. Note that $\theta(x_n) = z_n$, since the definition of $\theta(x_n)$ is just z_n given by (20) with $\tilde{f}$-trajectory starting at x_n. So (18) holds. The continuity of θ follows from the continuity of τ^{cu} and τ^u resulting from (10) and (12), and from the local boundedness of $C(x)$, and therefore from the local uniform convergence of $z_0 = \theta(x) = \lim_{j\to\infty} \tilde{\Phi}^{-j}\tau^u\tau^{cu}\tilde{f}^j(x)$ in (20).

For $x \in G^u$ we have $C(x)$ bounded by a constant $C(\epsilon)$ depending only on ϵ, and $\tau^u\tau^{cu}(x) = x$. Hence $\rho(\theta(x), x)$ is bounded, by $C(\epsilon)C_\lambda\frac{1}{1-\lambda^{-1}\xi}$. Therefore, by the continuity, θ maps G and even G^u, onto G^u, by topological reasons. Indeed, for any y and the ball $B(e,r)$ of sufficiently large radius r the map $\theta : \partial B(e,r) \to G^u \setminus \{y\}$ is homotopic in $G^u \setminus \{y\}$ to the identity on $\partial B(e,r)$. If θ on $B(e,r)$ omitted y, the homotopy would extend to $B(x,r)$ yielding retraction of $\operatorname{cl} B(e,r)$ to it boundary. Consequently for any y the equation $\theta(x) = y$ has a solution. Finally (19) holds with $C = C(\epsilon)C_\lambda\frac{1}{1-\lambda^{-1}\xi}$. ∎

Proof of Theorem 2.2, i.e. inequality (3). Since $\tilde{f}$ is a lift of f on a compact M, it is uniformly continuous, hence it has a modulus of continuity function. Denote it by $\mu_{\tilde{f}}$. Set

$$\mathbf{a} = \inf\{\rho(xg, x) : g \in \Gamma \setminus \{e\},\, x \in G\}\,.$$

Since the right invariant metric ρ need not be left invariant, $\mathbf{a}$ can be less than $\inf\{\rho(g,e) : g \in \Gamma \setminus \{e\}\}$. Fortunately $\mathbf{a} > 0$ since $G \to G/\Gamma$ is a covering map. (Here is a formal proof: $x = yh$ where $h \in \Gamma$ and $\rho(y,e) \leq \operatorname{diam} M$, the diameter in ρ. Hence $\rho(xg,x) = \rho(yhg,yh) = \rho(yhgh^{-1}y^{-1},e)$, so $\mathbf{a} \geq \inf\{\rho(yky^{-1},e) : k \in \Gamma \setminus \{e\}, \rho(y,e) \leq \operatorname{diam} M\}$. It is nonzero because only a finite number of k's count in the infimum. If for example $\rho(yky^{-1},e) \leq 1$ then for $h = yky^{-1}$ we have $k = y^{-1}hy$ so $\rho(k,e) \leq 2\operatorname{diam} M + 1$.)

Choose $\delta > 0$ such that

$$\delta + \mu_{\tilde{f}}(\delta) < \mathbf{a}.$$

There exists $p \in G^u$ such that $U := \theta(B(p,\delta/2))$ has nonempty interior in G^u by the Baire Theorem. Indeed, we cover G^u by a countable number of balls $B_j = B(g_j,\delta/2)$ and if every $\theta(B_j)$ have empty interior then $\bigcup_j \theta(B_j)$ is of the first category, what contradicts the property that $\theta|_{G^u}$ maps onto G^u. (In the sequel it is sufficient to know that $\operatorname{vol}(U) > 0$.)

For each $\xi > 1$ and n large enough, there exists for Φ an (n,ξ^{2n})-separated set $S_n \subset U$ such that

$$\sharp S_n \geq \operatorname{vol}(U) 2^{-u} \xi^{-2nu} \prod_{j=1}^{u} |\lambda_j|^n, \tag{22}$$

where u is the dimension of E^u and λ_j are all the eigenvalues of $D\Phi(e)$ of absolute value larger than 1, each counted with its multiplicity. This can be seen by a volume argument. Indeed in the invariant measure (volume) induced by our Riemannian metric

$$\operatorname{vol}(\Phi^n(U)) = \operatorname{Jacobian}(\Phi^n|_U)\operatorname{vol}(U) \geq \operatorname{vol}(U) \prod_{j=1}^{u} |\lambda_j|^n.$$

Let A be a maximal ξ^{2n}-separated subset of $\Phi^n(U)$ (that is $\rho(x,y) \geq \xi^{2n}$ for all distinct $x,y \in A$). Then, by the maximality of A, $\bigcup_{x\in A} B(x,\xi^{2n}) \supset \Phi^n(U)$. Hence

$$\sum_{x\in A} \operatorname{vol} B(x,\xi^{2n}) \geq \operatorname{vol}(U) \prod_{j=1}^{u} |\lambda_j|^n.$$

Finally, apply

$$\operatorname{vol} B(x,\xi^{2n}) = \operatorname{vol} B(e,\xi^{2n}) = \operatorname{Const} \xi^{2nu},$$

where Const $\leq 2^u$ is the volume of the unit ball in the Lie algebra E^u. The first equality holds since the metric is right invariant. The second equality holds since

$\exp(B(0,r)) = B(e,r)$ (a standard fact) and therefore we have $\operatorname{vol} B(e,r) = \int_{B(0,r)} \text{Jacobian}(\exp(X))dX$. But for $X \in \mathcal{G}$, $D(\exp)(X) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)!} (\operatorname{ad}_X)^i$ (cf., [8] Part I., Chapter 2.3.3) and consequently $\det D(\exp)(X) = 1$, because ad_X is nilpotent. Now we conclude the proof of (22), where we set $S_n = \tilde{\Phi}^{-n}(A)$.

For each $x \in S_n$ we choose an arbitrary $x' \in \theta^{-1}(x) \cap B(p, \delta/2)$ in G^u. We shall prove that the set $\{x'\Gamma \in M : x \in S_n\}$ is (n,δ)-separated for f.

Suppose to the contrary that there exist $x, y \in S_n$ with $x \neq y$, such that for x', y' as above $\rho(f^k(x'\Gamma), f^k(y'\Gamma)) < \delta$ for all $l = 0, \dots, n$. We prove by induction that $\rho(\tilde{f}^l(x'), \tilde{f}^l(y')) < \delta$. First note that $\rho(x', y') < \delta$ by definition (both points belong to $B(p, \delta/2)$). Suppose we know that $\rho(\tilde{f}^l(x'), \tilde{f}^l(y')) < \delta$ for an integer $l < n$. Then $\rho(\tilde{f}^{l+1}(x'), \tilde{f}^{l+1}(y')) < \mu_{\tilde{f}}(\delta)$. By our supposition there exists $h \in \Gamma$ such that

$$\rho(\tilde{f}^{l+1}(x'), \tilde{f}^{l+1}(y')h) = \rho(f^{l+1}(x'\Gamma), f^{l+1}(y'\Gamma)) < \delta\,.$$

So, $\rho(\tilde{f}^{l+1}(y'), \tilde{f}^{l+1}(y')h) \leq \delta + \mu_{\tilde{f}}(\delta) < \mathbf{a}$, hence $h = e$ by the definition of $\mathbf{a}$. In consequence $\rho(\tilde{f}^{l+1}(x'), \tilde{f}^{l+1}(y')) < \delta$. The induction procedure is finished.

Now let us apply θ. We argue as at the beginning of Proof of Proposition 2.10. By (14) in Lemma 2.7 $r(\tau^s(\tilde{f}^n(x')) \leq \Delta_1$ (and the same for y'), with Δ_1 depending on $\epsilon = \sup \rho(\tilde{f}, \Phi)$ and $\epsilon_1 = 0$. Hence by (10) in Lemma 2.6

$$\rho(\tau^{cu}\tilde{f}^n(x'), \tau^{cu}\tilde{f}^n(y')) \leq \delta' := \mu(C\mu(\delta)(1 + r(\Delta_1) + \cdots + \frac{1}{(k-1)!} r(\Delta_1)^{k-1})).$$

Next, by (16) in Lemma 2.8, for ξ replaced by $\xi_1 := \xi^{1/k}$

$$\tau^c \tau^{cu} \tilde{f}^n(x') \leq C\xi_1^n$$

and the same for y', for C depending on $\xi_1, \epsilon' = \Delta_2$ and $\epsilon'_1 = 0$. So, by Lemma 2.6 (12)

$$\begin{aligned}\rho(\tau^u\tau^{cu}\tilde{f}^n(x'), \tau^u\tau^{cu}\tilde{f}^n(y')) &\leq C\mu(\delta')(1 + C\xi_1^n + \cdots + \frac{1}{(k-1)!}(C\xi_1^n)^{k-1}) \\ &\leq \operatorname{Const} \xi_1^{nk}\,.\end{aligned}$$

By (19) in Proposition 2.10,

$$\rho(\tau^u\tau^{cu}\tilde{f}^n(x'), \Phi^n(x)) \leq C\xi^n \quad \text{and} \quad \rho(\tau^u\tau^{cu}\tilde{f}^n(y'), \Phi^n(y)) \leq C\xi^n\,.$$

Hence $\rho(\Phi^n(x), \Phi^n(y)) \leq \operatorname{Const} \xi^n$. This contradicts $\rho(\Phi^n(x), \Phi^n(y)) \geq \xi^{2n}$, see the definitions of A and S_n.

Note finally that $\log\prod_{j=1}^{u}|\lambda_j| = \log \mathrm{sp}\,(\wedge D\Phi(e))$, compare (1) in Introduction. Hence, passing with n to ∞ we get from (22) the estimate

$$\log \mathrm{sp}\,(\wedge D\Phi_f(e)) \leq \mathbf{h}(f) + 2u\log\xi.$$

Letting in the construction $\xi \searrow 1$, we conclude the proof of the Theorem 2.2, hence the proof of Theorem 1.3. ∎

Remark 2.11: We expect that Theorem 1.3 holds for G exponential solvable. A natural additional assumption would be that E^c is contained in the nilradical $\mathcal{N}$. Under this assumption however, assuming also that Φ is an automorphism, G must be nilpotent! See S. Smale [30, Proposition 3.6] and N. Bourbaki [2] Exercise §4 21b for the hyperbolic and more general — no roots of unity — case, and Exercise 11 to deal with the $E^c \subset \mathcal{N}$ case.

Remark 2.12 (Proof of Theorem 1.4): In the case $E^c \subset \mathcal{N}$ and Φ is an endomorphism, but not an automorphism, G need not be a nilmanifold. However Entropy Conjecture holds, provided G is of type (R). Split $\mathcal{G}$ into a subspace E^0 corresponding to the eigenvalue 0 for $D\Phi(e)$ and to the subspace E^+ corresponding to all other eigenvalues, see the definitions preceding Proposition 2.3. Note that by Remark 2.11 $E^+ \subset \mathcal{N}$, thus we are in the situation of Theorem 1.4.

To prove Theorem 1.4 recall that by Proposition 2.3 E^0 is an ideal. Project first by τ^+ to the group $G^+ = \exp E^+$, in the decomposition $G = G_1G_2$ for $G_1 = G^0 = \exp E^0$ and $G_2 = G^+$. For an $\tilde{f}$-trajectory x_n being an ϵ-Φ-trajectory, we write $x_{n+1} = h_1h_2g_1g_2$ for $\Phi(x_n) = g_1g_2$. In our convention, all the elements of G written with index 1 belong to G_1, and written with index 2 belong to G_2. By the normality of G^0 we can write $x_{n+1} = h_1g_1'h_2g_2$, hence $\rho(\tau^+(x_{n+1}, \Phi(\tau^+(x_n)) = r(h_2) \leq \mu(\rho(x_{n+1}, \Phi(x_n))) \leq \mu(\epsilon)$. Since E^+ is contained in the nilradical, i.e., E^+, G^+ are nilpotent we can proceed next as in Proof of Theorem 2.2. We project by τ^{cu} and τ^u and shadow, as before.

Remark 2.13: The proof given here has followed the procedure applied for $M = \mathbb{T}^m$ in [21] and in an early (unpublished) version of [23] by M. Misiurewicz. It has been crucial that $\theta(G)$ has nonempty interior, or at least nonzero measure induced by the Riemann metric restricted to $W^u(e)$. For a more detailed discussion see Remark 4.8.

3. Proof of Lemma 2.7.

1. THE METRICS. All $\mathcal{G}_j$ in the descending central tower of ideals, see (4) in the Introduction, are Φ-invariant, hence each $\mathcal{G}_j, j = 0, \ldots, k-1$ is spanned by $E^s \cap \mathcal{G}_j$ and $E^{cu} \cap \mathcal{G}_j$. Therefore, for each j, one can choose subspaces $E^s_j \subset ((\mathcal{G}_j \setminus \mathcal{G}_{j+1}) \cup \{0\}) \cap E^s$ and $E^{cu}_j \subset ((\mathcal{G}_j \setminus \mathcal{G}_{j+1}) \cup \{0\}) \cap E^{cu}$ spanning, together with $\mathcal{G}_{j+1}$, the space $\mathcal{G}_j$. We assume about our Riemannian metric ω that all E^s_j and E^{cu}_i are pairwise orthogonal.

For an arbitrary $a : 0 < a < 1$ we shall consider also the metric ω_a preserving the spaces E^s_j and E^{cu}_i orthogonal, but multiplying ω on each $\hat{\mathcal{G}}_j := E^s_j \oplus E^{cu}_j$ by a_j^2, where $a_j := a^{3^j}$.

Note that

$$\|[X,Y]\|_a \le aC_\omega \|X\|_a \|Y\|_a \,, \tag{23}$$

where we simplify the indexing by replacing ω_a by a.

Indeed, writing $X = \sum_{j=0}^{k-1} X_j$, $Y = \sum_{j=0}^{k-1} Y_j$ where $X_j, Y_j \in \hat{\mathcal{G}}_j$, we get

$$\|[X,Y]\|_a \le \sum_{s,t} \|[X_s, Y_t]\|_a \le \sum_{s,t} a_{u(s,t)} \|[X_s, Y_t]\| \,,$$

where $u = u(s,t) > \max\{s,t\}$ denote the minimal integer such that $[X_s, Y_t] \subset \mathcal{G}_u$. Continuing, we get

$$\begin{aligned}
\|[X,Y]\|_a &\le \sum_{s,t} a_u C_\omega \|X_s\| \|Y_t\| = \sum_{s,t} a_u C_\omega a_s^{-1} \|X_s\|_a a_t^{-1} \|Y_t\|_a \\
&\le aC_\omega \sum_{s,t} \|X_s\|_a \|Y_t\|_a \\
&\le aC_\omega k^2 \sup_s \|X_s\|_a \sup_t \|Y_t\|_a \\
&\le aC_\omega k^2 \|X\|_a \|Y\|_a.
\end{aligned}$$

Summarizing

$$\|[X,Y]\|_a \le C_a \|X\|_a \|Y\|_a, \quad \text{with } C_a = ak^2 C_\omega$$

arbitrarily small when a is small appropriately. Remark that we can always get C arbitrarily small by multiplying a Riemannian metric by a large constant. Here, however, we achieved C small not by just increasing the metric. Namely, here

$$\omega_a \le \omega \,.$$

The constant a will depend on ϵ, as will be specified later on.

2. THE STRATEGY. For an ϵ-Φ-trajectory x_n we have $x_{n+1} = h_1h_2\Phi(x_n)$ for $h_1 \in G^s, h_2 \in G^{cu}$ with $r_a(h_i) \leq r(h_i) \leq \mu(\epsilon)$.

Write $\Phi(x_n) = g_1g_2$ for $g_1 \in G^s, g_2 \in G^{cu}$. We have $x_{n+1} = h_1h_2g_1g_2$ and the problem is with changing the order of h_2 and g_1. Let us write

$$h_1h_2g_1g_2 = h_1g_1(g_1^{-1}h_2g_1h_2^{-1})h_2g_2 \,.$$

We shall proceed now in 2 steps:

1. Estimate $r_a(z)$, where z denotes the commutator $z = g_1^{-1}h_2g_1h_2^{-1}$.
2. Estimate $r_a(k_i)$, $i = 1, 2$ for the decomposition $z = k_1k_2$, $k_1 \in G^s$, $k_2 \in G^{cu}$.

3. THE COMMUTATOR. First let us differentiate with respect to t the function (curve) $z(t) = x_tyx_t^{-1}y^{-1}$ for any $x_t = \exp tX$, $y = \exp Y$, $X, Y \in \mathcal{G}$. At any t_0,

$$\begin{aligned}(24)\qquad & d/dt|_{t_0}(x_tyx_t^{-1}y^{-1}) \\ &= d/dt|_0(x_t)(x_{t_0}yx_{t_0}^{-1}y^{-1}) + x_{t_0}y(-d/dt|_0(x_t))(x_{t_0}^{-1}y^{-1}) \\ &= X(z(t_0)) - \mathrm{Ad}_{x_{t_0}}\mathrm{Ad}_y(X)(z(t_0)) \\ &= (X - e^{\mathrm{ad}\,_{t_0X}}e^{\mathrm{ad}\,_Y}(X))(z(t_0)) \\ &= -\sum_{n_1\geq 0, n_2 \geq 1} \frac{1}{n_1!n_2!}[(t_0X)^{n_1}[Y^{n_2}X]](z(t_0))\,,\end{aligned}$$

where Y^{n_2} means the n_2-th iterate of $\mathrm{ad}\,_Y$, similarly $(t_0X)^{n_1}$. We have used the convention that multiplying a vector tangent to G by an element of G from the right or left hand side means acting on the vector by the differential of the action of this multiplication on G.

Therefore,

$$r_a(z(1)) \leq \int_0^1 \|d/dt(z(t))\|_a dt \leq \sum_{n_1\geq 0, n_2\geq 1} \frac{1}{n_1!n_2!} C_a^{n_1+n_2}\|X\|_a^{n_1+1}\|Y\|_a^{n_2}.$$

If we assume that $\|X\|_a, \|Y\|_a \leq R$ for a constant $R > 0$, then we conclude with

$$(25)\qquad r_a(z(1)) \leq Re^{C_aR}(e^{C_aR} - 1) \leq R(1 + 2C_aR)2C_aR \leq 1/2$$

for a small enough (depending on R).

4. The decomposition. Given $X, Y \in \mathcal{G}$ we have, by Campbell–Hausdorff formula, $\mathrm{Log}\,(\exp X \exp Y) = X + Y + [X,Y] + \cdots$. Hence,

$$\|\mathrm{Log}\,(\exp X \exp Y) - (X+Y)\|_a \le C_a \cdot \|X\|_a \cdot \|Y\|_a \cdot W(C_a, \|X\|_a, \|Y\|_a),$$

where W is a polynomial determined by the Campbell–Hausdorff formula and the nilpotency of G. Taking absolute values of the standard coefficients we can assume that all the coefficients of W are nonnegative. Assume that X is orthogonal to Y in ω_a and $\|X\|_a, \|Y\|_a \le 1$. Then

$$\begin{aligned}\|\mathrm{Log}\,(\exp X \exp Y) - (X+Y)\|_a &\le C_a \max\{\|X\|_a, \|Y\|_a\} W(C_a, 1, 1)\\ &< \frac{1}{2} \max\{\|X\|_a, \|Y\|_a\}\end{aligned}$$

for a small enough.

In particular, $\mathrm{Log}\,(\exp X \exp Y)$ maps the "sphere"

$$S := \{X + Y : \max\{\|X\|_a, \|Y\|_a\} = 1\}$$

to $\mathcal{G} \setminus \{Z_0\}$ with degree 1 for every $Z_0 \in \{\|Z\|_a \le 1/2\}$, since $\mathrm{Log}\,(\exp X \exp Y) : S \to \mathcal{G} \setminus \{Z_0\}$ is homotopic to the identity. In conclusion

$$\{\mathrm{Log}\,(\exp X \exp Y) : \|X\|_a \le 1, \|Y\|_a \le 1\} \supset \{Z \in \mathcal{G} : \|Z\|_a \le 1/2\}, \tag{26}$$

compare the end of Proof of Proposition 2.10.

5. Conclusion. Now we can conclude the proof of Lemma 2.7. Let $\lambda : 0 < \lambda < 1$ and $A > 0$ satisfy

$$\|D\Phi(e)^n(X)\| \le A\lambda^n \|X\| \quad \text{for all } X \in E^s \text{ and } n \ge 0\,, \tag{27}$$

compare (6) in Proposition 2.4. Define, for k the length of the descending central tower $\mathcal{G}_j$,

$$R := \lambda k A\Big(\frac{\mu(\epsilon) + 1}{1 - \lambda} + \epsilon_1\Big). \tag{28}$$

Adjust a to this R so that (25) holds, and small enough that (26) holds. Observe that (27) holds for the metric ω_a with the same constant λ and A replaced by kA. To see this decompose $X = \sum X_j$ with $X_j \in E^s_j$ and use the fact that each $E^s \cap \mathcal{G}_j$ is invariant under $D\Phi(e)$, the construction of ω_a and the assumption that $a \le 1$.

In $x_n = h_1 g_1 z h_2 g_2$, with the commutator $z = k_1 k_2$ as above, write $h_1 = a_n, k_1 = b_n, h_2 = c_n, z = d_n$. Recall that $g_1 = \Phi(\tau_1(x_{n-1}))$. So we can write

$$\tau_1(x_n) = a_n \Phi(\tau_1(x_{n-1})) b_n.$$

Introducing this notation for all n we get

$$\Phi(\tau_1(x_{n-1})) = \Phi(a_{n-1}) \cdots \Phi^{n-1}(a_1) \Phi^n(\tau_1(x_0)) \Phi^{n-1}(b_1) \cdots \Phi(b_{n-1}),$$

hence, knowing that $r_a(a_j) \le r(a_j) \le \mu(\epsilon)$ for all j, and assuming (the inductive assumption) that $r_a(b_j) \le 1$ for all $j = 1, \dots, n-1$, we can write

$$r_a\bigg(\Phi(\tau_1(x_{n-1})) \le R_n := kA\bigg(\sum_{j=1}^{n-1} \lambda^j \mu(\epsilon) + \lambda^n r_a(\tau_1(x_0))\bigg) + \sum_{j=1}^{n-1} \lambda^j\bigg) \le R\,.$$

Hence, using also $r_a(c_n) \le R$, which we achieve taking, say, A large enough, we can apply (25) and (26) getting $d_n = \exp X \exp Y$ for $\|X\|_a, \|Y\|_a \le 1$. Hence $r_a(b_n) \le 1$ for $b_n = \exp X$, which finishes the inductive step.

We conclude with all $r_a(\Phi(\tau_1(x_n)) \le R_{n+1} < R$, hence $r_a(\tau_1(x_n)) \le \mu(\epsilon) + R + 1$, hence with $r(\tau_1(x_n)) \le \Delta_1 := \text{Const} \cdot R$ with R defined by (28), where, say, Const $= 2\lambda^{-1} \sup \| \cdot \| / \| \cdot \|_a = 2\lambda^{-1} a^{-3^{k-1}}$. Finally Δ_2 can be computed from (10) in Lemma 2.7. ∎

4. Final discussion. A proof via Lefschetz and Nielsen numbers

We begin this section with a further information and examples of solvmanifolds.

For $m \ge 3$ and any ring $\mathcal{R}$ with unity we denote by $N_m(\mathcal{R})$ the group of all unipotent upper triangular $m \times m$ matrices with entries in $\mathcal{R}$. The nilmanifolds $N_m(\mathbb{R})/N_m(\mathbb{Z})$ are called **the Iwasawa manifolds**. Also, $N_n(\mathbb{C})/N_n(\mathbb{Z}[\imath])$, where $\mathbb{Z}[\imath]$ is the ring of Gaussian integers, is a nilmanifold. The three dimensional Iwasawa nilmanifold $N_3(\mathbb{R})/N_3(\mathbb{Z})$ is called baby-nil, and it is the simplest nonabelian nilmanifold. Lie algebra of $N_3(\mathbb{R})$ is called the Heisenberg Lie algebra.

We show now how to get the estimate (2), and (3), but under an additional assumption, as a consequence of already known theorems.

We start with the following result of Hattori [11] for a solvmanifold of type (R) that generalized a previous result of Nomizu [24] for nilmanifolds. We recall that for a given Lie algebra $\mathcal{G}$ the Chevalley–Eilenberg complex $(\Lambda^*\mathcal{G}^*, \delta)$ associated with $\mathcal{G}$ consists of the exterior algebra $\Lambda^*\mathcal{G}^*$ of the dual space $\mathcal{G}^*$

considered as a complex of vector spaces with the j-th, $0 \leq j \leq m$ gradation equal to $\wedge^j \mathcal{G}^*$ and the differential $\delta : \wedge^j \mathcal{G}^* \to \wedge^{j+1} \mathcal{G}^*$ defined as

$$\delta(X^*)(X_1, \ldots, X_{j+1}) := \sum_{1 \leq s \leq t \leq j+1} (-1)^{s+t-1} X^*([X_s, X_t], X_1, \ldots, \hat{X}_s, \ldots, \hat{X}_t, \ldots, X_{j+1}) \,.$$

THEOREM 4.1: *Let $(\Lambda^* \mathcal{G}^*, \delta)$ denote the Chevalley–Eilenberg complex associated to the Lie algebra $\mathcal{G}$ of a simply connected Lie group G of type (R). If $\Gamma \subset G$ is a discrete uniform subgroup, then $H^*(G/\Gamma; \mathbb{R}) \cong H^*(\Lambda^* \mathcal{G}^*, \delta)$.*

Note that the Chevalley–Eilenberg complex can be identified with a subcomplex of de Rham complex consisting of, say, right invariant forms. This result together with the Hopf formula (see [31]) leads to the following.

PROPOSITION 4.2: *Let $f : G/\Gamma \to G/\Gamma$ be a self map of a compact special solvmanifold of type (R) and $\Phi_f : G \to G$ be an endomorphism associated to f. Then for the linear operator $D\Phi_f(e)$ we have the inclusion of spectra $\sigma(H^*(f)) \subset \sigma(\wedge D\Phi_f(e))$ and consequently the estimate $\mathrm{sp}\,(f) \leq \mathrm{sp}\,(\wedge D\Phi_f(e))$. Moreover, for the Lefschetz number we have $L(f^n) = \det(\mathbb{I} - (D\Phi_f(e))^n)$ for every $n \in \mathbb{N}$.*

Proof. By the Nomizu–Hattori theorem the spectral radius and Lefschetz number of f can be derived by use of the map $D\Phi_f(e)^*$ of the Chevalley–Eilenberg complex. Since $D\Phi_f(e)$ is a homomorphism of the Lie algebra, the linear subspaces of co-boundaries, co-cycles are preserved by $\wedge D\Phi_f(e)^*$. Consequently, the cohomology spaces can be identified with the factors of subspaces preserved by $\wedge D\Phi_f(e)^*$. The inclusion, and consequently the inequality, follows from the fact that the spectrum of an operator restricted to an invariant subspace is a subset of the spectrum of entire operator and the same is true for the factors. Finally we have $L(f^n) = \sum_{k=0}^{k=m} (-1)^k \mathrm{tr}\, H^k(f^n) = \sum_{k=0}^{k=m} (-1)^k \mathrm{tr}\,, \bigwedge^k (D\Phi_f(e)^n)^* = \det(\mathbb{I} - (D\Phi_f(e))^n)$, where m is the dimension of G and the second equality is a direct consequence of the Hopf formula and linear algebra. ∎

Now we show that the linear map $D\Phi_f(e)$ has the same spectrum as an integer $m \times m$-matrix A_f assigned to a self-map f of compact nilmanifold M, of dimension m, called the linearization of f and used in the Nielsen theory approach (cf., [17, 13]).

To do this we need some more information on the nilmanifolds. Let G be a nilpotent group and let

$$\{e\} = G_k \lhd G_{k-1} \lhd G_{k-2} \lhd \cdots \lhd G_1 \lhd G_0 = G$$

be the central tower of normal subgroups $G_i := [G, G_{i-1}] = \exp(\mathcal{G}_i)$ corresponding to the central tower (4) of the Lie algebra. Then for any uniform lattice Γ in G, each group of a descending tower of normal (in Γ) discrete subgroups:

$$\{e\} = \Gamma_k \lhd \Gamma_{k-1} \lhd \Gamma_{k-2} \lhd \cdots \lhd \Gamma_1 \lhd \Gamma_0 = \Gamma \ , \tag{29}$$

$\Gamma_i := \Gamma \cap G_i$ is a uniform lattice in the corresponding subgroup G_i of the central tower (cf., [19] and [26]). Note that every homomorphism $\Phi : G \to G$ preserving Γ, induces factor maps ϕ_i, of the corresponding factor manifolds $B_i := (G_i/\Gamma_i)/(G_{i+1}/\Gamma_{i+1}) = (G_i/G_{i+1})\big/(\Gamma_i/\Gamma_{i+1})$. Since G_i/G_{i+1} is abelian and $\Gamma_i \subset G_i$ is uniform, the manifold B_i is a torus of dimension m_i. Consequently ϕ_i is a torus endomorphism and thus represented by an $m_i \times m_i$ integer matrix.

Definition 4.3 (cf. [17, 13]): Let $M = G/\Gamma$ be a compact nilmanifold of dimension m, $f : M \to M$ be a continuous self-map, and $\Phi_f : G \to G$, an endomorphism associated to f. We define an integer $m \times m$ matrix A_f as the direct sum

$$A_f = \bigoplus_{i=0}^{k-1} A_i \ ,$$

where each A_i is $m_i \times m_i$ integer matrix of the endomorphism $\phi_i : \mathbb{T}^{m_i} \to \mathbb{T}^{m_i}$.

Note that A_f is a homotopy invariant, and if $M = \mathbb{T}^m$ is a torus then A_f is equal to the matrix of the endomorphism $f_\# : \mathbb{Z}^m \to \mathbb{Z}^m$ induced on the fundamental group.

Proposition 4.4: *Let $f : M \to M$ be a map of a nilmanifold. Then we have $\sigma(D\Phi_f(e)) = \sigma(A_f)$. Consequently* $\operatorname{sp}(D\Phi_f(e)) = \operatorname{sp}(A_f)$ *and* $\operatorname{sp}(\bigwedge D\Phi_f(e)) = \operatorname{sp}(\bigwedge A_f)$.

Proof. To shorten notation put $D := D\Phi_f(e)$. If $M = \mathbb{T}^m$, then $G = \mathbb{R}^m$ and Φ is a linear map preserving a uniform lattice Γ, we can identify G and its commutative algebra $\mathcal{G}$, and A_f is the matrix of D in the basis of $\mathcal{G}$ formed by generators of Γ.

In the general case note that D preserves the central tower of ideals (4) as an endomorphism of the Lie algebra $\mathcal{G}$. Note that the quotient Lie algebra of G/G_{k-1} is equal to $\mathcal{G}/\mathcal{G}_{k-1}$. Consider in $\mathcal{G}$ the linear space basis, formed by generators of Γ_{k-1} in $\mathcal{G}_{k-1}$ identified with $G_{k-1} = \mathbb{R}^{m_{k-1}}$ and preimages in the factorization $\mathcal{G} \to \mathcal{G}/\mathcal{G}_{k-1}$ of any basis in $\mathcal{G}/\mathcal{G}_{k-1}$. Then the matrix of D has the form

$$D = \begin{bmatrix} D_{k-1} & * \\ 0 & \tilde{D}_{k-1} \end{bmatrix},$$

where $D_{k-1} = D|_{\mathcal{G}_{k-1}}$ and $\tilde{D}_{k-1} = D\tilde{\Phi}_f(e_{k-1})$, the factor of D to $\mathcal{G}/\mathcal{G}_{k-1}$, the differential at the unit element of G/G_{k-1} for $\tilde{\Phi}_f$ the factor of Φ_f to G/G_{k-1}.

Consequently for the characteristic polynomials we have that $\chi_D(t) = \chi_{D_{k-1}}(t)\,\chi_{\tilde{D}_{k-1}}(t)$, and the statement follows by the induction argument. ∎

Proof of Theorem 1.5. First note that if $f : M \to M$ is a map of a compact special solvmanifold of type (R), then $N(f) = 0$ is equivalent to the fact that f is homotopic to fixed point free map. The implication in one direction follows from the property of Nielsen number. Conversely, if $f : M \to M$ is a map of a compact manifold and $\dim M \geq 3$, then the Wecken theorem says that $N(f) = 0$ implies that f is deformable to a fixed point free map (see [14] for a modern proof of the Wecken theorem). If M is a special compact solvmanifold of type (R) of dim ≤ 2 it must be the torus and the statement holds by elementary analysis (cf., [13]).

Definition 4.5: The asymptotic Nielsen number is $N^\infty(f) := \limsup_n \sqrt[n]{N(f^n)}$, where $N(f)$ is the Nielsen number of f (cf., [14]).

In [12] Ivanov showed that

$$\log N^\infty(f) \leq \mathbf{h}(f), \tag{30}$$

for every continuous self-map f of a compact manifold. By the Anosov theorem (cf., [1], [4]) we have $N(f) = |L(f)|$ for a map of a compact nilmanifold, or a compact NR-solvmanifold (cf., [17]). From this and Proposition 4.2 it follows that $N(f^n) = |L(f^n)| = |\det(\mathrm{I} - D\Phi^n)|$, for $D\Phi := D\Phi(e)$. On the other hand $\det(\mathrm{I} - D\Phi^n) = 1 - \sum_{j=1}^m \lambda_j^n + \sum\limits_{j_1<j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n \lambda_1^n \lambda_2^n \cdots \lambda_m^n$, where

$\{\lambda_1, \dots, \lambda_m\}$ are all the eigenvalues of $D\Phi$, counted with multiplicities. This shows that

$$\begin{aligned}\log N^\infty(f) &\\ &= \limsup_{n\to\infty} \frac{1}{n} \log\Big(\Big|1-\sum_1^m \lambda_j^n + \sum_{j_1<j_2} \lambda_{j_1}^n \lambda_{j_2}^n + \cdots + (-1)^n \lambda_1^n \lambda_2^n \cdots \lambda_m^n\Big|\Big) \\ &= \begin{cases} -\infty & \text{if } 1 \in \sigma(D\Phi(e)), \\ \log\big(\prod_{|\lambda_j|>1} |\lambda_j|\big) = \sum_{|\lambda_j|>1} \log|\lambda_j| & \text{otherwise,} \end{cases}\end{aligned}$$

since we have assumed that $\{i : |\lambda_i| > 1\} \neq \emptyset$ (see (1)). To prove the latter case equality note that for $D^c := D\Phi(e)|_{E^c}$ the restriction of $D\Phi(e)$ to the central subspace, where $|\lambda_i| = 1$, we have the estimate $\limsup_{n\to\infty} |\det(I-(D^c)^n)| > 0$. To estimate this limsup consider for example the subsequence $n_k = 1 + kW$, where $W = \prod_i p_i$ for positive integers p_i such that $\lambda_i^{p_i} = 1$ for all $\lambda_i \in \sigma(D^c)$. By the unique ergodicity of irrational rotations of the circle, for each $\lambda = e^{2\pi i t_\lambda} \in \sigma(D^c)$ with t_λ irrational, the density of the set of k's such that λ^{1+kW} hits the arc α of the length $2\pi A$ containing 1 is equal to A. Therefore, if there are T irrational t_λs, the density of the set of the integers k so that all λ^{1+kW} omit α is at least $1 - AT$ which is positive if $A < 1/T$. If $T = 0$, then $\lambda^{1+kW} = \lambda$, so λ^{1+kW} have arguments bounded away from 0 for all $\lambda = \lambda_i$ and k.

Consequently we get $\log N^\infty(f) = \log \mathrm{sp}\,(\wedge D\Phi_f(e))$, see (1), provided $1 \notin \sigma(D\Phi_f(e))$. Theorem 2.1, the rigidity property, and the inequality (30) give the statement of Theorem 1.5. ∎

It is natural to ask whether the tools used for the proof of Theorem 1.5 work in a more general situation. We discuss it below.

Remark 4.6 (Behavior of the asymptotic Nielsen number): Every compact special solvmanifold $M = G/\Gamma$ can be represented (in many ways, in general, depending on a group $\tilde{N} \subset N$, where N is the nilradical) as a fibration $M_{\tilde{N}} \subset M \xrightarrow{p} \mathbb{T}^n$, with the fiber $M_{\tilde{N}}$ is a nilmanifold and the base $\mathbb{T}^n$ being the r-dimensional torus. Each such fibration is called the Mostow fibration (cf., [8] for the construction and more details). Due to the McCord theorem (see [17] for references) every map $f : M \to M$ can be deformed to a fiber preserving map $f = (f_{M_{\tilde{N}}}, f_{\mathbb{T}})$ of this fibration provided $f_\#(\Gamma \cap \tilde{N}) \subset (\Gamma \cap \tilde{N})$, for the map induced on fundamental group. The later condition is satisfied if as a nilpotent group $\tilde{N}$ we take $[G,G]$. Then one can define the linearization matrix A_f as $A_{f_{M_{\tilde{N}}}} \oplus A_{f_{\mathbb{T}}}$. But only for a map of the so-called NR-solvmanifold (a class a little bit larger than exponential manifolds) the equality $\det(\mathrm{I} - A_f) = L(f)$ and Anosov theorem hold (cf., [17]). Consequently, only for these solvmanifolds we get the estimate of $N^\infty(f)$ by $\mathrm{sp}\,(\bigwedge(A_f))$ by already known facts.

Remark 4.7 (Inequality $\mathrm{sp}\,(f) \leq \mathrm{sp}\,(\wedge(A_f))$): For the proof of this inequality we used the rigidity property replacing f by ϕ_f. On the other hand $f : M \to M$ is always, up to homotopy, a fiber map of the Mostow fibration. Using the Serre spectral sequence convergent to $H^*(M;\mathbb{R})$ one can show that $\mathrm{sp}\,(f) \leq \mathrm{sp}\,(E_2^{p,q}(f))$, because at each step passing from $E_r^*(M)$ to $E_{r+1}^*(M)$, $r \geq 2$, and then from $E_{r_0}^*(M) = E_\infty^*(M)$ to $H^*(M)$ we either pass to a subspace or to a factor space. To get directly $\mathrm{sp}\,(E_2^{p,q}(f)) = \mathrm{sp}\,(f_{M_N})\cdot\mathrm{sp}\,(f_{\mathbb{T}})$, we need to assume that the fibration is orientable, i.e. that the system of local coefficients of this fibration is constant (cf., [31]). Indeed $E_2^{p,q} = H^p(\mathbb{T}) \otimes H^q(M_N)$, which yields the discussed inequality. Unfortunately the Mostow fibration is not orientable in general.

Summing up, the statement of Theorem 1.5 still holds for a map f of a compact special NR-solvmanifold G/Γ provided f is homotopic to the factor of an endomorphism Φ of G, satisfying $1 \notin \sigma(D\Phi(e))$.

Remark 4.8: In [21] A. Manning proposed for any M the following proof that

$$\mathbf{h}(f) \geq \log \mathrm{sp}^{\wedge}(f)\,, \tag{31}$$

where $\mathrm{sp}^{\wedge}(f)$ is computed for $H^{\wedge}(f)$ which is $H^*(f)$ restricted to the subalgebra $H^{\wedge}(M;\mathbb{R}) \subset H^*(M;\mathbb{R})$ generated by the first cohomology space $H^1(M;\mathbb{R})$. Let $\mathbb{T}^k$ be the torus of dimension k being the dimension of $H^1(M;\mathbb{R})$. Then there exists a mapping $q : M \to \mathbb{T}^k$ such that $q \circ f$ is homotopic to $\phi \circ q$, $H^1(q)$ is an isomorphism on the first real (co)homology space and ϕ is an endomorphism of $\mathbb{T}^k$ inducing $H^1(f)$. Then there exists $\theta : \tilde{M} \to E^{u'}$, such that $\theta \circ \tilde{f} = \Phi\theta$ (compare (18)) where $\tilde{M}$ is the universal cover of M, Φ is the endomorphism of $\mathbb{R}^k$ whose factor is ϕ and $E^{u'}$ is the linear Φ-invariant subspace of $\mathbb{R}^k$, of dimension u', contained in the unstable subspace E^u corresponding to all the eigenvalues of Φ of modulus larger than 1, of maximal growth of volume under Φ^n, representing an element ω in the cohomologies whose g^*-image in $H^*(M;\mathbb{R})$ is non-zero.

Unfortunately it is not clear (as pointed out by D. Fried and M. Gromov) that there exists a k-simplex in M, whose lift to $\tilde{M}$ and the image by θ has nonzero measure in $E^{u'}$. See, in particular, [7, §8] and [9, Appendix 5].

Manning's proof of (31) is correct if $M = G/\Gamma$ is a nilmanifold, since then $q : M \to G/G_1/\Gamma/\Gamma_1 = \mathbb{T}^k$, as $G_1 = [G, G]$ is the commutator of G, and θ is "onto" by the same arguments as in Proof of Proposition 2.10.

Nevertheless, even if the above statement (31) is true it does not lead to a proof of Theorem 1.3 as follows from the following observation.

Proposition 4.9: *Let $H^\wedge(M)$ be the subalgebra of $H^*(M)$ generated (multiplicatively) by $H^1(M)$. For a compact nilmanifold $H^\wedge(M) = H^*(M)$ if and only if M is a torus.*

Proof. Since the cohomology algebra $H^*(\mathbb{T})$ is the exterior power algebra of $H^1(\mathbb{T})$, one implication is obvious.

Suppose now that $M = G/\Gamma$ is a nonabelian compact nilmanifold of dimension $m \geq 3$, i.e., $\Gamma = \pi_1(M)$ is non-abelian nilpotent group. The exact sequence of abelianization $e \subset [\Gamma, \Gamma] \subset \Gamma \to \Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}^k$ leads to a fibration of nilmanifolds $F \overset{i}{\subset} M \overset{p}{\to} \mathbb{T}^k$ corresponding to the homomorphism of fundamental groups. Here $k \geq 1$ and F is a compact nilmanifold of dimension $b = m - k \geq 1$. The Hurewicz theorem says that for every CW-complex $H_1(X; \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. Applying it to $X = M$ we get $H_1(M; \mathbb{Z}) = H_1(\mathbb{T}^k; \mathbb{Z}) = \mathbb{Z}^k$. Since M and $\mathbb{T}^k$ are $K(\pi, 1)$-spaces, this isomorphism is induced by p.

It means that $H_1(p) : H_1(M; \mathbb{Z}) \to H_1(\mathbb{T}^k; \mathbb{Z}) = \mathbb{Z}^k$ is an isomorphism, and so is $H_1(p) : H_1(M; \mathbb{R}) \to H_1(\mathbb{T}^k; \mathbb{R})$. Consequently the dual map $H^1(p) : H^1(\mathbb{T}^k; \mathbb{R}) \to H^1(M; \mathbb{R})$ is an isomorphism. This shows that $H^\wedge(M; \mathbb{R})$ is contained in the algebra generated by $p^*(H^*(\mathbb{T}^k; \mathbb{R})$. But the latter subalgebra vanishes in all dimensions $> k$, which shows that $H^\wedge(M; \mathbb{R})$ is then a proper subset of $H^*(M; \mathbb{R})$. ∎

Remark 4.10: Now we give an example that Theorem 1.3 gives sharper estimate that inequality (31). Let us take G the baby-nil $M = N_3(\mathbb{R})/N_3(\mathbb{Z})$, which is generated by the matrices

$$x = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with relations $xz = zx$, $yz = zy$, $xyx^{-1}y^{-1} = z$. The same matrices generate $N_3(\mathbb{Z})$. It is easy to verify that the mapping $\Phi(x) = x^{k_1}$, $\Phi(y) = y^{k_2}$, $\Phi(z) = z^{k_3}$, $k_i \in \mathbb{Z}$, extends to an endomorphism preserving Γ, provided $k_1 \cdot k_2 = k_3$. In the coordinates $X = \exp^{-1}(x)$, $Y = \exp^{-1}(y)$, $Z = \exp^{-1}(z)$ the matrix of $D\Phi(e)$ is diagonal, with the entries k_i on the diagonal. Using the argument of the proof of Proposition 4.9 we get that $\mathrm{sp}\,(H^\wedge(\phi)) = \mathrm{sp}\,(\tilde{\phi})$, where $\tilde{\phi} : \mathbb{T}^2 \to \mathbb{T}^2$ is the induced by Φ self-map of the base $(G/\Gamma)/([G,G]/[G,G]\cap\Gamma)$, where $[G,G] = Z(G) = \mathrm{span}\{z\}$, (subspace generated by z) of the corresponding fibration. Consequently, $\mathrm{sp}\,(H^\wedge(\phi)) = |k_1 k_2| < (k_1 k_2)^2 = \mathrm{sp}\,(\wedge D\Phi(e))$ if $|k_1\, k_2| > 1$. Note that from Theorem 4.13 it follows that $\mathbf{h}(\phi) = (k_1 k_2)^2$.

In [30, p. 762] S. Smale provided examples of Anosov diffeomorphisms ϕ of 6-dimensional $(G \times G)/\Gamma$, Γ a uniform lattice, being factors of automorphisms Φ of $G \times G$ for $G = N_3(\mathbb{R})$, also satisfying $\mathrm{sp}\,(H^\wedge(\phi)) < \mathrm{sp}\,(\wedge D\Phi(e))$.

Finally, we would like to discuss briefly a relation between the topological entropy and periodic points for a nilmanifold map. The claim that $\mathbf{h}(f) > 0$ implies a nonempty set of periodic points, or the set of minimal periods $\mathrm{Per}(f)$ of f is false in view of the following standard examples. Take $M = \mathbb{T}^1 \times \mathbb{T}^1 = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the product map $f = (f_1, f_2)$, $f : \mathbb{T}^2 \to \mathbb{T}^2$ where f_1 is of degree > 1; mapping of the circle $\mathbb{T}^1$ and f_2 has no periodic points. (There is even an example of a minimal homeomorphism of $\mathbb{T}^2$, hence without periodic points, with $\mathbf{h}(f) > 0$, by M. Rees. On the other hand, for f of class $C^{1+\epsilon}$, this cannot happen, see [10].) Conversely, for the map of the torus $f(x,y) = (x, x+y)$ mod 1 we have $\mathrm{Per}(f) = \mathbb{N}$ but $\mathbf{h}(f) = 0$.

It seems to be more natural to compare the behavior of the entropy with the set of so called homotopy minimal periods (see [13]).

Definition 4.11: Let $f : X \to X$ be a map of a topological space and $\mathrm{Per}(f)$ its set of minimal periods. We define

$$\mathrm{HPer}(f) =: \bigcap_{g \sim f} \mathrm{Per}(g) \subset \mathrm{Per}(f)$$

where the intersection is taken over all g homotopic to f, i.e., a natural number belongs to $\mathrm{HPer}(f)$ if and only if its a minimal period of all maps homotopic to f.

Corollary 4.12: *Let $f : M \to M$ be a map of compact nilmanifold. If* $\mathrm{HPer}(f)$ *is infinite then* $\mathbf{h}(f) > 0$.

Proof. From the result of [13] it directly follows that for a nilmanifold map the set $\mathrm{HPer}(f)$ is infinite if and only if $1 \notin \sigma(A_f)$ and $\mathrm{sp}\,(A_f) > 1$. By Theorem 4.4, $\mathrm{sp}\,(\wedge D\Phi(e)) = \mathrm{sp}\,(\bigwedge A_f)$. Then the claim follows either directly from Theorem 2.1, or from the Ivanov inequality (30), since $\mathrm{sp}\,(\bigwedge A) \geq \mathrm{sp}\,(A) > 1$. ∎

We end showing the following

Theorem 4.13: *Let $M = G/\Gamma$ be a quotient of a connected Lie group by a uniform lattice Γ and $\phi : M \to M$ be the factor map of a preserving Γ endomorphism $\Phi : G \to G$. Then $\mathbf{h}(\phi) = \log \mathrm{sp}\,(\bigwedge D\Phi(e))$. If M is a nilmanifold, then ϕ minimizes the entropy in its homotopy class.*

Proof. It follows from [27], [18] or [16] (the latter in the case of diffeomorphism), see also [25] , that $\mathbf{h}(f) \leq \limsup_{n\to\infty} n^{-1} \sup_{x\in M} \log \| \bigwedge (Df(x))\|$ for every C^1-mapping $f : M \to M$ of a compact manifold M. In our case $\| \bigwedge D\phi(x)\| = \| \bigwedge D\Phi(e)\|$ for every $x \in M = G/\Gamma$. Hence, by the definition of spectral radius, $\mathbf{h}(\phi) \leq \log \mathrm{sp}\,(\bigwedge D\Phi(e))$.

The opposite inequality can be proved as Theorem 2.2 but much simpler. Just choose separated sets S_n for Φ in a ball in G^u of origin at e. (In fact, we can consider S_n directly in G/Γ.)

Finally, since by Theorem 2.2, $\mathbf{h}(f) \geq \log \mathrm{sp}\,(\bigwedge D\Phi_f(e))$ in the nilmanifold case, $\mathbf{h}(\phi)$ minimizes $\mathbf{h}(f)$ in the homotopy class. ∎

Added in proof: The results have been recently extended by the authors to infra-nilmanifolds in particular in “Estimates of the topological entropy from below for continuous self-maps on some compact manifolds” Discrete and Continuous Dynamical Systems, 21.2 (2008).

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